

IMPROPER FILTRATIONS FOR C^* -ALGEBRAS: SPECTRA OF UNILATERAL TRIDIAGONAL OPERATORS

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To Béla Szökefalvi-Nagy on the occasion of his eightieth birthday

ABSTRACT. Let $\mathcal{A} \subseteq \mathcal{B}(H)$ be a C^* -algebra of operators and let $P_1 \leq P_2 \leq \dots$ be an increasing sequence of finite dimensional projections in $\mathcal{B}(H)$. In a previous paper [3] we developed methods for computing the spectrum of self adjoint operators $T \in \mathcal{A}$ in terms of the spectra of the associated sequence of finite dimensional compressions $P_n T P_n$. In a suitable context, we showed that this is possible when P_n increases to $\mathbf{1}$. In this paper we drop that hypothesis and obtain an appropriate generalization of the main results of [3].

Let $P_+ = \lim_n P_n$, $H_+ = P_+ H$. The set $\mathcal{A}_+ \subseteq \mathcal{B}(H_+)$ of all compact perturbations of operators $P_+ T \upharpoonright_{H_+}$, $T \in \mathcal{A}$, is a C^* -algebra which is somewhat analogous to the Toeplitz C^* -algebra acting on H^2 . Indeed, in the most important examples \mathcal{A} is a simple unital C^* -algebra having a unique tracial state, the operators in \mathcal{A} are “bilateral”, those in \mathcal{A}_+ are “unilateral”, and there is a short exact sequence of C^* -algebras

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{A}_+ \rightarrow \mathcal{A} \rightarrow 0$$

whose features are central to this problem of approximating spectra of operators in \mathcal{A} in terms of the eigenvalues of their finite dimensional compressions along the given filtration.

This work was undertaken in order to develop an efficient method for computing the spectra of discretized Hamiltonians of one dimensional quantum systems in terms of “unilateral” tridiagonal $n \times n$ matrices. The solution of that problem is presented in Theorem 3.4.

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1. Introduction.

There is a natural way to discretize the Hamiltonian of a one dimensional quantum system in such a way that a) the uncertainty principle is preserved, and b) the discretized operator is in a form appropriate for carrying out numerical studies [1,2]. In a suitable representation, this discretized Hamiltonian is a bilateral tridiagonal operator T of the form

$$1.1 \quad Te_n = e_{n-1} + v(\cos(n\theta))e_n + e_{n+1}$$

where $\{e_n : n \in \mathbb{Z}\}$ is an orthonormal basis for a Hilbert space H , and v is a real function in $C[-1, +1]$. v is a rescaled version of the potential of the system. The number θ is related to the numerical step size, and may properly be considered an irrational multiple of π [1].

The spectra of these operators cannot be computed explicitly. For example, even in the simplest case where $v(x) = 2x$ (in physical terms, the case of a one dimensional harmonic oscillator) it is not yet known if the spectrum of T is totally disconnected. Choi, Elliott and Lui have shown that this is the case when θ/π is a Liouville number [9], but despite significant progress since the early eighties this ‘Ten Martini’ problem of Mark Kac remains open in general [5,6,7,8,9,14]. To our knowledge, essentially nothing is known in the case of an arbitrary continuous potential v . Thus it is of interest to understand how one might carry out numerical calculations of the spectrum using established techniques for computing eigenvalues of self-adjoint $n \times n$ matrices, and to describe the precise sense in which these eigenvalues converge to the spectrum of T as $n \rightarrow \infty$. A general approach to this problem was worked out in [3].

Here we will consider operators T of the following somewhat more general tridiagonal form

$$1.2 \quad Te_n = e_{n-1} + d_n e_n + e_{n+1}$$

where $\{d_n : n \in \mathbb{Z}\}$ is a bounded bilateral sequence of real numbers which is *almost periodic but not periodic*. A sequence $d_n = v(\cos(n\theta))$ as in formula 1.1 will satisfy this hypothesis whenever θ/π is irrational and $v \in C[-1, +1]$ is not a constant. In [3], we showed that such a program will succeed for operators of the form 1.1 by establishing a formula which shows how the spectrum of T is determined by the eigenvalues of the sequence of $2n+1 \times 2n+1$ matrices

$$\begin{pmatrix} d_{-n} & 1 & 0 & \dots & 0 & 0 \\ 1 & d_{-n+1} & 1 & \dots & 0 & 0 \\ 0 & 1 & d_{-n+2} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & d_{n-1} & 1 \\ 0 & 0 & 0 & \dots & 1 & d_n \end{pmatrix}.$$

However, it is more convenient to compute with the sequence of “unilateral” $n \times n$ compressions of T

$$1.3 \quad T_n = \begin{pmatrix} d_1 & 1 & 0 & \dots & 0 & 0 \\ 1 & d_2 & 1 & \dots & 0 & 0 \\ 0 & 1 & d_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & d_{n-1} & 1 \end{pmatrix}.$$

In fact, the latter has been implemented for the Ten Martini operators of Mark Kac in a Macintosh program [4], and it is effective. In this paper we will prove that this procedure will *always* be successful by establishing a variation of the formula [3, formula 5.1] which is applicable to the sequence of unilateral compressions 1.3.

In this setting the formula takes the following form. Let T_n be as in 1.3, $n = 1, 2, \dots$ and let $\lambda_1^n < \lambda_2^n < \dots < \lambda_n^n$ be the eigenvalue list of T_n (recall that the eigenvalues are distinct because T_n is tridiagonal [12, p. 124]). Then there is a probability measure μ_T , whose closed support is precisely the spectrum of T , such that for every continuous function $f \in C_0(\mathbb{R})$ we have

$$1.4 \quad \lim_{n \rightarrow \infty} \frac{1}{n} (f(\lambda_1^n) + f(\lambda_2^n) + \dots + f(\lambda_n^n)) = \int_{-\infty}^{+\infty} f(x) d\mu_T(x).$$

It follows that a real number λ is in the spectrum of T iff the eigenvalues of the sequence of matrices T_1, T_2, \dots accumulate rapidly enough in any fixed neighborhood U of λ so that their density in U is positive. We also describe the measure μ_T in terms of the unique tracial state on a certain C^* -algebra associated with the operator T . See Theorem 3.4 and the remarks following it.

Formula 1.4 shows quite explicitly how one should go about calculating approximations to the measure μ_T , and it is these calculations that have been implemented in the program [4] cited above.

2. Improper filtrations.

In this section we analyze the situation described above in a rather abstract context, and establish an appropriate generalization of the main result of [3]. Let us first recall some terminology from [3]. Let H be a separable infinite dimensional Hilbert space. A *filtration* of H is an increasing sequence $H_1 \subseteq H_2 \subseteq \dots$ of finite dimensional subspaces of H with the property that $\cup_n H_n$ is dense in H .

Here, we will simply drop the latter condition on the sequence H_n . In order to avoid annoying trivialities, we do require that the closed subspace

$$H_+ = \overline{\cup_n H_n}$$

generated by the filtration should be infinite dimensional. The filtration is called *proper* or *improper* according as $H_+ = H$ or $H_+ \neq H$. Let P_n denote the projection onto H_n . As in [3] we may introduce the notion of the *degree* of an operator $T \in \mathcal{B}(H)$ relative to a filtration,

$$\deg(T) = \sup_n \operatorname{rank}(P_n T - T P_n).$$

The degree of an operator is a nonnegative integer or $+\infty$, and properties established in [3] such as $\deg(ST) \leq \deg(S) + \deg(T)$ persist in this more general context.

For example, if $\{e_1, e_2, \dots\}$ is an infinite orthonormal sequence in H and H_n is the linear span of $\{e_1, \dots, e_n\}$, then $\{H_1 \subseteq H_2 \subseteq \dots\}$ is a filtration which is typically improper. As we pointed out in [3] in the case of proper filtrations, any operator whose matrix relative to this orthonormal set is band-limited must have finite degree. Thus, *it is as appropriate here as it was in [3] to think of finite degree operators as abstractions of band-limited matrices.* Any self-adjoint operator T determines a multitude of filtrations relative to which it has degree 1:

choose any vector e in H such that $\{e, Te, T^2e, \dots\}$ is linearly independent and put $H_n = [e, Te, T^2e, \dots, T^{n-1}e]$. This filtration is proper iff e is a cyclic vector for T . Of course, in general there can be very irregular jumps in the dimensions of the subspaces of a filtration.

Finally, if \mathcal{A} is a C^* -algebra of operators acting on H and $\mathcal{F} = \{H_1 \subseteq H_2 \subseteq \dots\}$ is a filtration of H , then \mathcal{F} is called an \mathcal{A} -filtration if the set of all finite degree operators in \mathcal{A} is norm-dense in \mathcal{A} . If \mathcal{A} is generated by a set of finite degree operators, then \mathcal{F} is called an \mathcal{A} -filtration [3]. We begin with two propositions which concern compact perturbations.

Proposition 2.1. *Let $\mathcal{A} \subseteq \mathcal{B}(H)$ be a C^* -algebra, let $\{H_1 \subseteq H_2 \subseteq \dots\}$ be an \mathcal{A} -filtration and let P_+ be the projection onto H_+ . Then $P_+A - AP_+$ is compact for every $A \in \mathcal{A}$.*

proof. Since the finite degree operators in \mathcal{A} are norm-dense and $P_+A - AP_+$ is norm-continuous in A , it suffices to show that

$$\text{rank}(P_+A - AP_+) \leq \deg(A) < \infty$$

for every $A \in \mathcal{A}$ having finite degree. Since

$$\text{rank}(P_nA - AP_n) \leq \deg(A)$$

for every $n = 1, 2, \dots$ and since $P_nA - AP_n$ converges to $P_+A - AP_+$ in the strong operator topology, the proposition follows from the fact that the rank function is weakly lower semicontinuous on $\mathcal{B}(H)$.

Since we lack an appropriate reference for the latter, we sketch a short proof for completeness. The assertion is that for every positive integer k , the set of operators

$$\mathcal{S}_k = \{T \in \mathcal{B}(H) : \text{rank}(T) \leq k\}$$

is weakly closed. Let K be the Hilbert space antisymmetric tensor product of $k+1$ copies of H . A subspace M of H has dimension at most k iff

$$M^{\wedge k+1} = M \wedge M \wedge \dots \wedge M = 0.$$

Thus, an operator T belongs to \mathcal{S}_k iff

$$T\xi_1 \wedge T\xi_2 \wedge \dots \wedge T\xi_{k+1} = 0$$

for every choice of vectors $\xi_1, \xi_2, \dots, \xi_{k+1}$ in H . Using the fact that K is spanned by vectors of the form $\eta_1 \wedge \eta_2 \wedge \dots \wedge \eta_{k+1}$ we see that T belongs to \mathcal{S}_k iff for every set of $2k+2$ vectors $\xi_1, \dots, \xi_{k+1}, \eta_1, \dots, \eta_{k+1}$ in H , the polynomial $p(T)$ defined by

$$p(T) = \langle T\xi_1 \wedge T\xi_2 \wedge \dots \wedge T\xi_{k+1}, \eta_1 \wedge \eta_2 \wedge \dots \wedge \eta_{k+1} \rangle$$

is zero. Taking note of the inner product in K , we see that $p(T)$ is the determinant of the $(k+1) \times (k+1)$ matrix whose ij th term is $\langle T\xi_i, \eta_j \rangle$, and is therefore weakly continuous. Hence \mathcal{S}_k is weakly closed \square

If one is given a positive linear functional ρ on a C^* -algebra \mathcal{C} , then the GNS construction gives rise to a representation π_ρ of \mathcal{C} on a Hilbert space H_ρ . We will say that ρ is infinite dimensional if H_ρ is an infinite dimensional Hilbert space.

Proposition 2.2. *Let $\mathcal{A} \subseteq \mathcal{B}(H)$ be a unital C^* -algebra having a unique tracial state τ , and assume τ is infinite dimensional. Then $\tau(K) = 0$ for every compact operator $K \in \mathcal{A}$.*

proof. Let π be the representation of \mathcal{A} obtained via the GNS construction on τ . Then $M = \pi(\mathcal{A})''$ is a von Neumann algebra with the property that the natural extension $\tilde{\tau}$ of τ to M is a normal tracial state of M .

Notice that M is a factor. Indeed, if e is any nonzero central projection in M then note first that $t = \tilde{\tau}(e) > 0$. Indeed, by the Schwarz inequality, $\tilde{\tau}(e) = 0$ implies $\tilde{\tau}(aeb) = \tau(eba) = 0$ for every $a, b \in M$, and hence $e = 0$ because $\tilde{\tau}$ is a vector state of M which is associated with a cyclic vector. In order to show that $e = \mathbf{1}$, we define a tracial state ρ of \mathcal{A} by

$$\rho(A) = \frac{1}{t} \tilde{\tau}(\pi(A)e).$$

Because of uniqueness, we have $\rho = \tau$. From this, together with the fact that $\pi(\mathcal{A})$ is strongly dense in M , it follows that $\tilde{\tau}(ae) = t\tilde{\tau}(a)$ for every $a \in M$. By an argument similar to the one given above to show that $e \neq 0$, we find that $e - t\mathbf{1} = 0$, hence $e = t\mathbf{1}$. Thus $t = 1$ and $e = \mathbf{1}$.

We may conclude that M is a finite factor. Now let \mathcal{J} be the ideal of all compact operators in \mathcal{A} . \mathcal{J} is a C^* -algebra of compact operators, and hence it must decompose uniquely into a C^* -algebraic direct sum of elementary C^* -algebras

$$\mathcal{J} = \sum_k \mathcal{J}_k.$$

It suffices to show that $\tau(\mathcal{J}_k) = 0$ for every k . Contrapositively, assume $\tau(\mathcal{J}_k) \neq 0$ for some k . Then $\pi(\mathcal{J}_k)$ must be nonzero too, and its weak closure in M must be all of M since it is a nonzero weakly closed ideal in a factor. Because τ is an infinite dimensional state M is infinite dimensional and therefore so is \mathcal{J}_k . But an infinite-dimensional elementary C^* -algebra has no nonzero bounded traces. Hence the restriction of τ to \mathcal{J}_k must be zero \square

Remark. Perhaps it is worth pointing out that Proposition 2.2 becomes false if one deletes the hypothesis that τ is an infinite dimensional state. For example, choose any C^* -algebra of the form $A = B \oplus C$ where B is a unital C^* -algebra having no nonzero finite traces (such as a Cuntz algebra) and C is an $n \times n$ matrix algebra. Certainly A has a unique tracial state τ , and there is an obvious faithful representation of A which contains finite rank operators not annihilated by τ .

Throughout the remainder of this section, we assume that we are given a unital C^* -algebra \mathcal{A} which has a unique tracial state τ . Every self-adjoint operator $A \in \mathcal{A}$ gives rise to a probability measure μ_A on the real line by way of

$$\int_{-\infty}^{+\infty} f(x) d\mu_A(x) = \tau(f(A)), \quad f \in C_0(\mathbb{R}).$$

μ_A is called the *spectral distribution* of A [3, section 4]. If τ is faithful then the closed support of μ_A is precisely the spectrum of A . Let $\mathcal{F} = \{H_1 \subseteq H_2 \dots\}$ be a perhaps improper \mathcal{A} -filtration and let H_+ and P_+ have the meaning assigned above. The following result is a generalization of theorem 4.5 of [3] to the case of improper filtrations.

Theorem 2.3. *Let $\mathcal{A} \subseteq \mathcal{B}(H)$ be a C^* -algebra having a unique tracial state, let $\{H_1 \subseteq H_2 \subseteq \dots\}$ be an \mathcal{A} -filtration, and assume that the subspace H_+ has the following property*

$$2.4 \quad A \upharpoonright_{H_+} = \text{compact} \implies A = \text{compact},$$

for every $A \in \mathcal{A}$. Let μ_A be the spectral distribution of a self-adjoint operator $A \in \mathcal{A}$, and let $[a, b]$ be the smallest closed interval containing $\sigma(A)$. For each n , let $d_n = \dim H_n$ and let $\lambda_1^n, \lambda_2^n, \dots, \lambda_{d_n}^n$ be the eigenvalue list of $A_n = P_n A \upharpoonright_{H_n}$. Then for every $f \in C[a, b]$,

$$\lim_{n \rightarrow \infty} \frac{1}{d_n} (f(\lambda_1^n) + f(\lambda_2^n) + \dots + f(\lambda_{d_n}^n)) = \int_a^b f(x) d\mu_A(x).$$

Remarks. We will write \mathcal{K} and \mathcal{K}_+ for the respective algebras of compact operators on H and H_+ . Notice that the hypothesis 2.4 is satisfied for trivial reasons if the filtration is proper, and in that case Theorem 2.3 reduces to [3, Theorem 4.5]. 2.4 is also automatic when \mathcal{A} is simple. Indeed, if T is an operator in \mathcal{A} whose restriction to H_+ is compact, then T belongs to the kernel of the $*$ -homomorphism $\omega : \mathcal{A} \rightarrow \mathcal{B}(H_+)/\mathcal{K}_+$ given by $\omega(X) = P_+ X \upharpoonright_{H_+} + \mathcal{K}_+$. Since ω is not the zero homomorphism (H_+ is infinite dimensional), it must be injective by simplicity. Hence $T = 0$.

proof of Theorem 2.3. Let \mathcal{A}_+ be the set of operators in $\mathcal{B}(H_+)$ of the form

$$\mathcal{A}_+ = P_+ A \upharpoonright_{H_+} + K,$$

where $A \in \mathcal{A}$ and K is a compact operator on H_+ . We will show first that

2.5 \mathcal{A}_+ is a C^* -algebra algebra having a unique tracial state τ_+ .

2.6 τ_+ is related to the tracial state τ of \mathcal{A} by way of

$$\tau_+(P_+ A \upharpoonright_{H_+} + K) = \tau(A), \quad A \in \mathcal{A}, K \in \mathcal{K}_+.$$

If A and B are two operators in \mathcal{A} with A_+ and B_+ denoting their respective compressions to H_+ , then from Proposition 2.1 we see that

$$A_+ B_+ = P_{H_+} A B \upharpoonright_{H_+} + \text{compact}.$$

It follows that \mathcal{A}_+ is a unital $*$ -subalgebra of $\mathcal{B}(H_+)$ which contains all compact operators. Thus, to prove that \mathcal{A}_+ is a C^* -algebra, it suffices to show that its image in the Calkin algebra $\mathcal{B}(H_+)/\mathcal{K}_+$ of H_+ is norm-closed. But the map $\pi : \mathcal{A} \rightarrow \mathcal{B}(H_+)/\mathcal{K}_+$ defined by

$$\pi(A) = P_{H_+} A \upharpoonright_{H_+} + \mathcal{K}_+$$

is clearly a unital $*$ -homomorphism whose range is the image of \mathcal{A}_+ in $\mathcal{B}(H_+)/\mathcal{K}_+$. Since the range of π must be norm-closed, we conclude that \mathcal{A}_+ is norm-closed.

In order to prove 2.5, we first exhibit a tracial state of \mathcal{A}_+ . Let $\sigma : \mathcal{A} \rightarrow \mathcal{B}(H_+)/\mathcal{K}_+$ be the map defined by

$$\sigma(A) = P_+ A \upharpoonright_{H_+} + \mathcal{K}_+$$

By Proposition 2.1, σ is a $*$ -homomorphism of unital C^* -algebras, and its range is precisely the image of \mathcal{A}_+ in the Calkin algebra. By 2.4, the kernel of σ is $\mathcal{A} \cap \mathcal{K}$; indeed $\sigma(A) = 0$ iff $\sigma(A^*A) = 0$ iff the restriction of A to H_+ is compact iff A is compact. Hence σ determines an isomorphism $\dot{\sigma}$ of C^* -algebras

$$\dot{\sigma} : \mathcal{A}/\mathcal{A} \cap \mathcal{K} \rightarrow \mathcal{A}_+/\mathcal{K}_+.$$

On the other hand, because of Proposition 2.2, τ vanishes on $\mathcal{A} \cap \mathcal{K}$ and thus can be promoted to a tracial state $\dot{\tau}$ on the quotient $\mathcal{A}/\mathcal{A} \cap \mathcal{K}$ by way of $\dot{\tau}(A + \mathcal{A} \cap \mathcal{K}) = \tau(A)$. Thus the composition $\dot{\tau} \circ \dot{\sigma}^{-1}$ defines a tracial state of $\mathcal{A}_+/\mathcal{K}_+$. If we now define τ_+ on \mathcal{A}_+ by $\tau_+(A) = \dot{\tau} \circ \dot{\sigma}^{-1}(A + \mathcal{K}_+)$, then τ_+ is certainly a tracial state on \mathcal{A}_+ . After unravelling its definition we find that τ is related to τ_+ by formula 2.6

$$2.7 \quad \tau_+(P_+A \upharpoonright + \mathcal{K}_+) = \tau(A), \quad A \in \mathcal{A}.$$

It remains to show that τ_+ is the only tracial state on \mathcal{A}_+ . To see that, let ρ be a tracial state on \mathcal{A}_+ . It is clear that the restriction of ρ to \mathcal{K}_+ must be zero, because there are no nonzero bounded traces on the algebra of compact operators on an infinite dimensional Hilbert space. Hence we may promote ρ to a tracial state $\dot{\rho}$ on $\mathcal{A}_+/\mathcal{K}_+$ by

$$\dot{\rho}(T + \mathcal{K}_+) = \rho(T), \quad T \in \mathcal{A}_+.$$

Making use of the isomorphism $\sigma : \mathcal{A}/\mathcal{A} \cap \mathcal{K} \rightarrow \mathcal{A}_+/\mathcal{K}_+$ exhibited above, we define a state ρ_0 on \mathcal{A} by

$$2.8 \quad \rho_0(A) = \dot{\rho} \circ \sigma(A + \mathcal{A} \cap \mathcal{K}) = \rho(P_+A \upharpoonright_{H_+} + \mathcal{K}_+).$$

ρ_0 is clearly a trace on \mathcal{A} , hence $\rho_0 = \tau$. 2.7 and 2.8 together show that ρ is the trace τ_+ exhibited above. Hence both assertions 2.5 and 2.6 are established.

It remains to deduce Theorem 2.3 from these two assertions and the results of [3]. For that, we view $\mathcal{F}_+ = \{H_1 \subseteq H_2 \subseteq \dots\}$ as a *proper* filtration of H_+ . Notice that if A is any finite degree operator in \mathcal{A} and B is an operator in \mathcal{A}_+ of the form $B = P_+A \upharpoonright_{H_+} + F$ where F is a finite rank operator on H_+ , then for every $n = 1, 2, \dots$ we have $P_nB - BP_n = P_+(P_nA - AP_n) \upharpoonright_{H_+} + P_nF - FP_n$. Hence

$$\text{rank}(P_nB - BP_n) \leq \sup_n \text{rank}(P_nA - AP_n) + 2\text{rank}(F) \leq \deg(A) + 2\text{rank}(F) < \infty.$$

It follows that the degree of B is finite. Such operators B are norm-dense in \mathcal{A}_+ since the finite degree operators of \mathcal{A} are norm dense in \mathcal{A} . We conclude that \mathcal{F} is a *proper \mathcal{A} -filtration of H_+* .

Thus we are in position to apply [3, Theorem 4.5] to the operator $A_+ = P_+A \upharpoonright_{H_+}$, the C^* -algebra \mathcal{A}_+ and the filtration \mathcal{F} . From [3, Theorem 4.5] we conclude that for every $f \in C_0(\mathbb{R})$,

$$2.9 \quad \lim_{n \rightarrow \infty} \frac{1}{d_n} (f(\lambda_1^n) + f(\lambda_2^n) + \dots + f(\lambda_{d_n}^n)) = \tau_+(f(A_+)).$$

From Proposition 2.1 and the fact that f is continuous we have

$$f(A_+) = P_+f(A) \upharpoonright_{H_+} + \text{compact}.$$

Using the formula 2.6, we can rewrite the right side of 2.9 as follows

$$\tau_+(P_+f(A) \upharpoonright_{H_+} + \text{compact}) = \tau(f(A)) = \int_a^b f(x) d\mu_A(x),$$

which establishes Theorem 2.3 \square

3. Applications.

In order apply the general results of section 2 to establish the formula 1.4, we recall some of the lore of almost periodic sequences of complex numbers. Consider the commutative C^* -algebra $l^\infty = l^\infty(\mathbb{Z})$. The additive group \mathbb{Z} acts naturally on l^∞ by translations, giving a C^* -dynamical system. An element $a \in l^\infty$ is *almost periodic* if the set of all of \mathbb{Z} -translates of a is a relatively norm-compact subset of l^∞ . The set $AP(\mathbb{Z})$ of all almost periodic sequences is a unital C^* -algebra-subalgebra of l^∞ which is invariant under the action of \mathbb{Z} .

The characters of \mathbb{Z} are the sequences

$$e_\lambda(n) = \lambda^n, \quad n \in \mathbb{Z}$$

where λ is a complex number of absolute value 1. Every e_λ is almost periodic, and in fact $AP(\mathbb{Z})$ is precisely the closed linear span of all characters. More generally, if S is any translation-invariant linear subspace of $AP(\mathbb{Z})$, then the *spectrum* of S is defined as the following subset of the unit circle

$$\sigma(S) = \{\lambda \in \mathbb{T} : e_\lambda \in S\}.$$

$\sigma(S)$ is nonempty if $S \neq 0$, and more generally every norm-closed translation-invariant subspace S of almost periodic sequences obeys uniform spectral synthesis in that

$$3.1 \quad S = \overline{\text{span}} \{e_\lambda : \lambda \in \sigma(S)\}.$$

If, in addition, S is a unital C^* -algebra subalgebra of $AP(\mathbb{Z})$ then $\sigma(S)$ is a subgroup of the circle group \mathbb{T} . Typically, $\sigma(S)$ fails to be closed in the usual topology of \mathbb{T} ; indeed, every subset of \mathbb{T} arises as the spectrum of some S (so long as one is working with an underlying set theory that is appropriate for functional analysis). It follows easily from this summary that *an almost periodic sequence a is periodic iff $\sigma(a)$ generates a finite subgroup of \mathbb{T} .*

Finally, the norm-closed convex hull of the set of translates of any almost periodic sequence a contains exactly one constant sequence $M(a)\mathbf{1}$, and the scalar $M(a)$ is the von Neumann mean of a :

$$M(a) = \lim_{n \rightarrow \infty} \frac{1}{2n+1} (a_{-n} + a_{-n+1} + \cdots + a_n).$$

Moreover, the convergence to $M(a)$ is *uniform* in that

$$\sup_{k \in \mathbb{Z}} |M(a) - \frac{1}{2n+1} (a_{k-n} + a_{k-n+1} + \cdots + a_{k+n})| \rightarrow 0$$

as $n \rightarrow \infty$. M defines a translation-invariant state of the C^* -algebra $AP(\mathbb{Z})$. We may conclude from these facts that if C is any unital translation-invariant C^* -subalgebra of $AP(\mathbb{Z})$, then the restriction of M to C is the *only* translation-invariant state on C . A convenient reference for most of the above is [11].

The following result is a consequence of standard techniques in operator algebras, together with the preceding observations. Since it is the essential link between the applications and the material of section 2, we include a proof.

Proposition 3.2. *Let $d = \{d_n : n \in \mathbb{Z}\}$ be a bounded real sequence which is almost periodic but not periodic and let D be the corresponding diagonal operator acting on $l^2(\mathbb{Z})$. Then the C^* -algebra generated by D and the bilateral shift is a simple C^* -algebra possessing a unique tracial state.*

proof. Let U be the bilateral shift acting on $l^2(\mathbb{Z})$, $Uf(n) = f(n-1)$, $n \in \mathbb{Z}$. For $f \in l^2(\mathbb{Z})$ we will write M_f for the obvious diagonal operator. Let \mathcal{A} be the C^* -algebra generated by $M_d = D$ and U . We first give a description of \mathcal{A} which is more convenient for our purposes.

Let Δ be the translation-invariant C^* -subalgebra of $AP(\mathbb{Z})$ generated by d and the constant sequence 1, and let \mathcal{D} be the set of all diagonal operators M_f , $f \in \Delta$. Since $U^n M_f U^{-n} = M_{f_n}$ where f_n is the translate of f by n , it follows that \mathcal{A} is the norm-closure of the set of all operators which are finite sums of the form

$$T = D_{-n}U^{-n} + D_{-n+1}U^{-n+1} + \cdots + D_nU^n$$

where each D_k belongs to \mathcal{D} and $n = 1, 2, \dots$. \mathcal{A} is therefore the image of the C^* -algebraic crossed product $\mathbb{Z} \times \Delta$ under the obvious representation it has on $l^2(\mathbb{Z})$. Thus, to show that \mathcal{A} is simple it suffices to show that $\mathbb{Z} \times \Delta$ is simple. By the characterization of simplicity of discrete crossed products given in [13, Theorem 8.11.12], it is enough to show that

- (1) the only closed translation-invariant nonzero ideal in Δ is Δ itself.
- (2) The Connes spectrum of the action of \mathbb{Z} on Δ is all of \mathbb{T} .

(1) follows immediately from the preceding remarks. Indeed, the preceding remarks imply that any translation-invariant ideal must contain enough unitary elements to span itself. Hence if it is nonzero it must be all of Δ . To prove (2), note that since (1) is valid and Δ is abelian, the Connes spectrum is identical with the Arveson spectrum, and the latter is simply the closure in \mathbb{T} of the subgroup

$$\sigma(\Delta) = \{\lambda \in \mathbb{T} : e_\lambda \in \Delta\}.$$

But since Δ contains the sequence d which is *not* periodic, $\sigma(\Delta)$ cannot be a finite subgroup of \mathbb{T} ; hence its closure is \mathbb{T} .

It remains to show that the crossed product $\mathbb{Z} \times \Delta$ has a unique tracial state. Indeed, the translation-invariant state of Δ gives rise to a tracial state τ of $\mathbb{Z} \times \Delta$, and therefore of \mathcal{A} . On the other hand, suppose that ρ is an arbitrary tracial state on \mathcal{A} . For each $\lambda \in \sigma(\Delta)$, consider the unitary operator $V_\lambda = M_{e_\lambda}$. V is a unitary representation of the discrete abelian group $\sigma(\Delta)$, \mathcal{D} is spanned by $\{V_\lambda : \lambda \in \sigma(\Delta)\}$, and moreover V satisfies the following commutation relation with U

$$V_\lambda U = \lambda U V_\lambda, \quad \lambda \in \sigma(\Delta).$$

This implies that for every $n \in \mathbb{Z}$ and $D \in \mathcal{D}$ we have

$$\rho(DU^n) = \rho(V_\lambda DU^n V_{-\lambda}) = \lambda^n \rho(DU^n),$$

for all $\lambda \in \sigma(\Delta)$. If $n \neq 0$ then it follows that $\rho(DU^n) = 0 = \tau(DU^n)$. If $n = 0$, then the restriction of ρ to \mathcal{D} gives rise to a translation-invariant state of Δ and hence it agrees with the restriction of τ to \mathcal{D} . Hence $\rho = \tau$. \square

We can now deduce applications to the problem of computing the spectra of discretized Hamiltonians. Let $d = \{d_n : n \in \mathbb{Z}\}$ be an almost periodic sequence of reals which is not periodic. Consider the Hilbert space $H = l^2(\mathbb{Z})$, let $\{e_n : n \in \mathbb{Z}\}$ be the obvious basis for H , and let T be the tridiagonal operator defined by 1.2. Let $\mathcal{A} \subseteq \mathcal{B}(H)$ be the C^* -algebra generated by the diagonal operator M_d and the bilateral shift. According to 3.2, \mathcal{A} is a simple C^* -algebra with a unique tracial state τ . T certainly belongs to \mathcal{A} , and we may consider its spectral distribution μ_T , defined by

$$3.3 \quad \int_{-\infty}^{+\infty} f(x) d\mu_T(x) = \tau(f(T)), \quad f \in C_0(\mathbb{R}).$$

Theorem 3.4. *The spectrum of T is the closed support of μ_T . Moreover, for every positive integer n let $\lambda_1^n < \lambda_2^n < \dots < \lambda_n^n$ be the eigenvalue list of the symmetric $n \times n$ matrix*

$$3.5 \quad \begin{pmatrix} d_1 & 1 & 0 & \dots & 0 & 0 \\ 1 & d_2 & 1 & \dots & 0 & 0 \\ 0 & 1 & d_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & d_{n-1} & 1 \\ 0 & 0 & 0 & \dots & 1 & d_n \end{pmatrix}.$$

Then for every $f \in C_0(\mathbb{R})$ we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} (f(\lambda_1^n) + f(\lambda_2^n) + \dots + f(\lambda_n^n)) = \int_{-\infty}^{+\infty} f(x) d\mu_T(x).$$

proof. Since \mathcal{A} is a simple C^* -algebra, τ must be a faithful trace. Thus it is apparent from definition 3.3 that μ_T must be supported precisely on the spectrum of T .

Let $\mathcal{F} = \{H_1 \subseteq H_2 \subseteq \dots\}$ be the filtration of H defined by $H_n = [e_1, e_2, \dots, e_n]$. Then \mathcal{F} is an improper filtration with respect to which the bilateral shift has degree 1 and every diagonal operator has degree 0. Hence \mathcal{F} is an \mathcal{A} -filtration. The hypothesis 2.4 of Theorem 2.3 is automatic in this case because of the simplicity of \mathcal{A} . Thus Theorem 3.4 follows after an application of Theorem 2.3 and its accompanying remarks \square

Remarks. We reiterate some remarks from [3] which reveal the significance of Theorem 3.4 for computations. For every $n = 1, 2, \dots$ and any Borel set $S \subseteq \mathbb{R}$, let $N_n(S)$ be the number of eigenvalues of the matrix 2.5 which belong to S . N_n is a positive integer-valued measure having total mass n . Theorem 3.5 asserts that the sequence of probability measures $n^{-1}N_n$ converges to μ_T in the weak*-topology of $C_0(\mathbb{R})$. Notice that this is the same notion of convergence that one has in the central limit theorem, and one may apply similar interpretations in the context of this paper.

For example, suppose that λ is a point in the spectrum of T . Then for every open neighborhood I of λ whose endpoints have μ_T -measure zero, and for any positive numbers α, β satisfying $\alpha < \mu_T(I) < \beta$ we have

for sufficiently large n . Notice that since α and β may be chosen arbitrarily close to $\mu_T(I)$, we have a precise estimate of the rate of growth of $N_n(I)$ with n . If, on the other hand, λ is not in the spectrum of T , then for a sufficiently small interval I containing λ we will have

$$\lim_{n \rightarrow \infty} \frac{1}{n} N_n(I) = \mu_T(I) = 0.$$

Actually, in this case one can say more, namely that the numbers $N_1(I), N_2(I), \dots$ are *uniformly bounded*. Indeed, since \mathcal{A} is simple it cannot contain any nonzero compact operators, and hence the spectrum of T is the same as its essential spectrum. The assertion follows from [3, Theorem 3.8]. In light of the fact that there are extremely fast algorithms for calculating the value of $N_n(S)$ for large but fixed values of n whenever S has the form $S = (-\infty, x]$ for $x \in \mathbb{R}$, these remarks provide the basis for a very efficient method of calculating the spectrum of T [4].

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